

RESTRICTION OF REPRESENTATIONS OF METAPLECTIC $\mathrm{GL}_2(F)$ TO TORI

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ABSTRACT. Let F be a non-Archimedean local field. We study the restriction of an irreducible admissible genuine representations of the two fold metaplectic cover $\widetilde{\mathrm{GL}}_2(F)$ of $\mathrm{GL}_2(F)$ to the inverse image in $\widetilde{\mathrm{GL}}_2(F)$ of a maximal torus in $\mathrm{GL}_2(F)$.

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1. INTRODUCTION

Let F be a non-Archimedean local field. A well-known theorem due to J. Tunnell [Tun83] for $p \neq 2$, and H. Saito [Sai93] in general, describes the restriction of an irreducible admissible representation of $\mathrm{GL}_2(F)$ to a maximal torus $E^\times \subset \mathrm{GL}_2(F)$, where E is any maximal commutative semisimple subalgebra of $M_2(F)$. One of the first conclusions about this restriction is that for any irreducible admissible representation π of $\mathrm{GL}_2(F)$ and a character $\chi : E^\times \rightarrow \mathbb{C}^\times$ such that $\chi|_{F^\times}$ is the central character of π , then

$$\dim \mathrm{Hom}_{E^\times}(\pi, \chi) \leq 1.$$

If π is a principal series representation of $\mathrm{GL}_2(F)$, or $E = F \oplus F$ and π not one dimensional, then we have

$$\dim \mathrm{Hom}_{E^\times}(\pi, \chi) = 1.$$

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This result on restriction of representations of $\mathrm{GL}_2(F)$ to maximal tori may be considered as the first case of branching laws from $\mathrm{SO}_{n+1}(F)$ to $\mathrm{SO}_n(F)$ which were formulated as conjectures by B. Gross and D. Prasad [GP92], and which were recently proved by Waldspurger and Mœglin-Waldspurger [MW12].

The aim of the present work is to initiate a similar study on restriction of representations of $\widetilde{\mathrm{GL}}_2(F)$, the metaplectic $\mathrm{GL}_2(F)$, which is a twofold cover of $\mathrm{GL}_2(F)$, to the inverse image \tilde{E}^\times of E^\times in $\widetilde{\mathrm{GL}}_2(F)$ where E is any maximal commutative semisimple subalgebra of $M_2(F)$.

We will see that one of the crucial first steps, that of multiplicity one, is lost in the metaplectic case, although there is still finiteness, even boundedness of multiplicities by explicit constants. It is hoped that metaplectic restriction problem will have some interest, and that this paper can serve as a first step.

The main theorem of this paper is the following.

Theorem 1.1. *Let E be any maximal commutative semisimple subalgebra of $M_2(F)$. Let π be an irreducible admissible genuine representation of $\widetilde{\mathrm{GL}}_2(F)$ with ω_π its central character (a character of $\tilde{F}^{\times 2}$). If E is a quadratic field extension of F , and π is supercuspidal, assume moreover that p , the residue characteristic of F , is odd. Then*

$$(1) \quad \pi|_{\tilde{E}^\times} \subseteq \mathrm{ind}_{\tilde{F}^{\times 2}}^{\tilde{E}^\times} \omega_\pi = \sum_{\sigma} (\dim \sigma) \sigma$$

where σ runs over all irreducible genuine representations of \tilde{E}^\times whose central character restricted to $\tilde{F}^{\times 2}$ is ω_π . Moreover if π is an irreducible principal series representation, then we have “equality” in (1).

Remark 1.2. *Of course we expect the theorem above to be true for $p = 2$ too which we are not able to achieve here. In the spirit of dichotomy of [GP92] we do not know if there is another representation π' of $\widetilde{\mathrm{GL}}_2(F)$ such that the restriction to \tilde{E}^\times of $\pi + \pi'$ achieves an “equality” up to finite error term in the above theorem, which is somehow accounted for by a twofold cover of D^\times (containing \tilde{E}^\times !), where D is the unique quaternion division algebra over F .*

Remark 1.3. *It will be seen later that for an irreducible genuine representation σ of \tilde{E}^\times , $\dim \sigma = |E^\times / F^\times E^{\times 2}|$, which for p odd equals 2 by Corollary 5.2, whereas for $p = 2$, by remark 5.3, $\dim \sigma = 2 \cdot 2^{\deg(F/\mathbb{Q}_2)}$.*

In particular, we see that the multiplicity of an irreducible genuine representation σ of \tilde{E}^\times in an irreducible admissible genuine representation of $\widetilde{\mathrm{GL}}_2(F)$ is at most $\dim \sigma$. The theorem turns out to be almost straightforward to prove in the cases when the representation π is either a principal series representation or $E = F \oplus F$. The more difficult part — something which we accomplish only for odd residue characteristic — is to understand the restriction of an irreducible genuine supercuspidal representation π of $\widetilde{\mathrm{GL}}_2(F)$ to \tilde{E}^\times where E/F is quadratic field extension and $E^\times \hookrightarrow \mathrm{GL}_2(F)$. In this case, when the residue characteristic is odd, we reduce the question on restriction from $\widetilde{\mathrm{GL}}_2(F)$ to \tilde{E}^\times to a question on restriction from $\mathrm{SL}_2(F)$ to E^1 , where E^1 is the group of norm 1 elements of E^\times . We shall do it in two steps.

- (1) Reduce the question on restriction from $\widetilde{\mathrm{GL}}_2(F)$ to \tilde{E}^\times to a question on restriction from $\widetilde{\mathrm{SL}}_2(F)$ to \tilde{E}^1 . This can be done for all residue characteristics.

- (2) When the residue characteristic is odd, using a correspondence (that we define in Section 8 using compact induction) between the set of isomorphism classes of irreducible genuine supercuspidal representation of $\widetilde{\mathrm{SL}}_2(F)$ and that of $\mathrm{SL}_2(F)$, we reduce the question of restriction from $\widetilde{\mathrm{SL}}_2(F)$ to \tilde{E}^1 to a question of restriction from $\mathrm{SL}_2(F)$ to E^1 .

In the second step, we need to restrict ourselves to the odd residue characteristic case because it is in this case when the metaplectic cover $\widetilde{\mathrm{SL}}_2(F)$ splits when it is restricted to a maximal compact subgroup of $\mathrm{SL}_2(F)$, and this splitting is used in executing the 2nd step.

We give a brief outline of the paper now. In Section 2, we recall the twofold cover of $\mathrm{GL}_2(F)$ under consideration. In Section 3, we describe the group structure on the inverse image in $\widetilde{\mathrm{GL}}_2(F)$ of maximal tori in $\mathrm{GL}_2(F)$ in an explicit way. The inverse images of tori may be called ‘Heisenberg groups’, which we discuss in some detail, proving some of its important properties and then describe their representations in Section 4. In Section 5, we prove that the inverse images of tori are Heisenberg groups in the sense as defined in the earlier section, and then we describe their irreducible genuine representations. In Section 6, the restriction of a genuine principal series representation to a non-split torus is considered. In Section 7, the restriction of any irreducible admissible genuine representation to the split torus has been considered. In Section 8, restricting ourselves to the case of odd residue characteristic, we define a correspondence between the irreducible genuine supercuspidal representation of $\widetilde{\mathrm{SL}}_2(F)$ and irreducible supercuspidal representation of $\mathrm{SL}_2(F)$. In Section 9, we study the restriction of an irreducible supercuspidal representation of $\widetilde{\mathrm{GL}}_2(F)$ to a non-split torus. We use the correspondence defined in Section 8 and transfer the question of this restriction to another question of restriction of a supercuspidal representation of $\mathrm{SL}_2(F)$ to E^1 .

In closing the introduction, we mention that [Pat15] which is the first author’s thesis, similar branching laws were considered from $\widetilde{\mathrm{GL}}_2(E)$ to $\mathrm{GL}_2(F)$ for E a quadratic extension of F , which may be considered as branching laws from SO_4 to SO_3 in the context of two-fold nonlinear covers. It may be added that the multiplicity formulae here involving $|E^\times / F^\times E^{\times 2}|$ (instead of 1 in [GP92]), as contained in remark 1.3, is also the multiplicity obtained in [Pat15] — and could well be considered to be true more generally for branching laws for twofold metaplectic covers of $\mathrm{GSpin}_n(F)$ to the corresponding cover of $\mathrm{GSpin}_{n-1}(F)$. Both the work [Pat15], and this one, has the common feature with [GP92], that to get this ‘uniform multiplicity’, we need to add *pure innerforms* of the smaller group in [Pat15], whereas here we need to add the contributions keeping the smaller group the same in this work. The methods in the two papers: [Pat15] and this one, are quite different.

2. PRELIMINARIES

Let F be a non-Archimedean local field. The group $\mathrm{SL}_2(F)$ has a unique twofold cover (up to isomorphism), called the metaplectic cover of $\mathrm{SL}_2(F)$ denoted by $\widetilde{\mathrm{SL}}_2(F)$. There are many in-equivalent twofold covers of $\mathrm{GL}_2(F)$ which extend the above twofold cover $\widetilde{\mathrm{SL}}_2(F)$ of $\mathrm{SL}_2(F)$. In what follows, we fix a twofold covering of $\mathrm{GL}_2(F)$

as follows. Note that $\mathrm{GL}_2(F) \cong \mathrm{SL}_2(F) \rtimes F^\times$ where $F^\times \hookrightarrow \mathrm{GL}_2(F)$ as $a \mapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$.

The action of F^\times on $\mathrm{SL}_2(F)$ lifts to an action on $\widetilde{\mathrm{SL}}_2(F)$. We fix the twofold cover $\widetilde{\mathrm{GL}}_2(F)$ of $\mathrm{GL}_2(F)$ as

$$\widetilde{\mathrm{GL}}_2(F) := \widetilde{\mathrm{SL}}_2(F) \rtimes F^\times$$

and call it the metaplectic cover of $\mathrm{GL}_2(F)$. We have a short exact sequence

$$1 \rightarrow \mu_2 \rightarrow \widetilde{\mathrm{GL}}_2(F) \rightarrow \mathrm{GL}_2(F) \rightarrow 1$$

where $\mu_2 = \{\pm 1\}$. This twofold cover $\widetilde{\mathrm{GL}}_2(F)$ of $\mathrm{GL}_2(F)$ is defined by a 2-cocycle, called Kubota cocycle

$$\beta : \mathrm{GL}_2(F) \times \mathrm{GL}_2(F) \rightarrow \mu_2.$$

We identify $\widetilde{\mathrm{GL}}_2(F)$ by $\mathrm{GL}_2(F) \times \mu_2$ as a set on which the group multiplication is defined using the cocycle β . Let B be the set of upper triangular matrices of $\mathrm{GL}_2(F)$. The restriction of β to B is given by

$$(2) \quad \beta \left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}, \begin{pmatrix} c & y \\ 0 & d \end{pmatrix} \right) = (a, d)_F$$

where $(\cdot, \cdot)_F$ denotes the quadratic Hilbert symbol of the field F . In particular, if $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $B = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$ and \tilde{A}, \tilde{B} are arbitrary lifts of A, B to $\widetilde{\mathrm{GL}}_2(F)$, we have

$$(3) \quad [\tilde{A}, \tilde{B}] = (a, d)_F (c, b)_F.$$

For a non-trivial character $\psi : F \rightarrow \mathbb{C}^\times$, let $\gamma(\psi)$ denote the 8-th root of unity associated to ψ by A. Weil, called the Weil index. For $a \in F^\times$, let $\psi_a : F \rightarrow \mathbb{C}^\times$ be the character of F given by $\psi_a(x) := \psi(ax)$. Define

$$\mu_\psi(a) := \frac{\gamma(\psi)}{\gamma(\psi_a)}.$$

It is known that

$$(4) \quad \mu_\psi(a)\mu_\psi(b) = (a, b)_F \mu_\psi(ab).$$

Let T_0 be the diagonal split torus of $\mathrm{SL}_2(F)$. Because of the commutation relation (3), the inverse image \tilde{T}_0 of T_0 is abelian. For $a \in F^\times$, let \underline{a} be the diagonal matrix $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in \mathrm{SL}_2(F)$. Because of (4), the map $\tilde{T}_0 \rightarrow \mathbb{C}^\times$ given by

$$(\underline{a}, \epsilon) \mapsto \epsilon \mu_\psi(a)$$

defines a genuine character of \tilde{T}_0 where $\epsilon \in \mu_2$.

For any subset X of $\mathrm{GL}_2(F)$, let \tilde{X} be the full inverse image inside $\widetilde{\mathrm{GL}}_2(F)$ determined by the projection $\widetilde{\mathrm{GL}}_2(F) \rightarrow \mathrm{GL}_2(F)$.

Recall that F^\times embedded diagonally as scalar matrices in $\mathrm{GL}_2(F)$ is the center of $\mathrm{GL}_2(F)$ and the covering $\widetilde{\mathrm{GL}}_2(F) \rightarrow \mathrm{GL}_2(F)$ restricted to the center of $\mathrm{GL}_2(F)$ is non-trivial. In fact, the cocycle is simply given by $\beta \left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \right) = (a, b)_F$ and hence the cover

$$1 \rightarrow \mu_2 \rightarrow \tilde{F}^\times \rightarrow F^\times \rightarrow 1$$

is non-trivial, although \tilde{F}^\times is an abelian group. Note that $(a, \epsilon) \mapsto \epsilon \mu_\psi(a)$, where $\epsilon \in \mu_2$, defines a genuine character of \tilde{F}^\times .

3. GROUP STRUCTURE OF INVERSE IMAGES OF THE TORI

Among the most important information about the covering $\widetilde{\mathrm{GL}}_2(F) \rightarrow \mathrm{GL}_2(F)$ for us is the precise knowledge about the group structure of the inverse image of the tori of $\mathrm{GL}_2(F)$ inside $\widetilde{\mathrm{GL}}_2(F)$. First consider the case of split torus.

Lemma 3.1. *Let T be the diagonal torus of $\mathrm{GL}_2(F)$ and $T^2 = \{t^2 : t \in T\}$. The subgroup \tilde{T}^2 is the center of the group \tilde{T} . Let $Z = F^\times$ denote the center of G , then the subgroup $\tilde{Z}\tilde{T}^2$ is a maximal abelian subgroup of \tilde{T} .*

Proof. From the commutation relation in (3), it is clear that \tilde{T}^2 is contained in the center. Let $x = \mathrm{diag}(a, b) \in T$ and $\tilde{x} \in \tilde{T}$ be any lift of x . If \tilde{x} is in the center of \tilde{T} then we prove that $a, b \in F^{\times 2}$. Suppose \tilde{x} is in the center of \tilde{T} . In particular, \tilde{x} commutes with $\mathrm{diag}(c, 1)$ and $\mathrm{diag}(1, d)$ for all $c, d \in F^\times$. By the commutation relation in (3), this implies $(c, b) = 1$ and $(a, d) = 1$ for all $c, d \in F^\times$, i.e. $a, b \in F^{\times 2}$. This proves that the center of \tilde{T} is \tilde{T}^2 . Since \tilde{Z} is abelian by the same commutation relation, $\tilde{Z}\tilde{T}^2$ is an abelian subgroup of \tilde{T} . We need to prove that it is a maximal abelian subgroup of \tilde{T} .

Take $\tilde{x} \in \tilde{T}$ as above, and suppose it commutes with all the elements $\tilde{y} \in \tilde{Z}\tilde{T}^2$ where $y = \mathrm{diag}(\alpha m^2, \alpha n^2)$ with $\alpha, m, n \in F^\times$. By the commutation relation in (3), we get that $(a, \alpha) = (b, \alpha)$, or in other words $(ab^{-1}, \alpha) = 1$ for all $\alpha \in F^\times$ and hence $ab^{-1} \in F^{\times 2}$. Thus $\tilde{x} \in \tilde{Z}\tilde{T}^2$. \square

Now we consider the case of a non-split torus. Let E/F be a quadratic extension. Let $E^\times \hookrightarrow \mathrm{GL}_2(F)$ be the non-split torus determined by the quadratic extension E/F . We will continue to denote by T the diagonal torus of $\mathrm{GL}_2(F)$.

Since the covering $1 \rightarrow \mu_2 \rightarrow \tilde{F}^\times \rightarrow F^\times \rightarrow 1$ is non-trivial and $F^\times \hookrightarrow E^\times$, the cover

$$1 \rightarrow \mu_2 \rightarrow \tilde{E}^\times \rightarrow E^\times \rightarrow 1$$

is also non-trivial. In fact, \tilde{E}^\times is a non-abelian group. The following lemma gives a more precise information on \tilde{E}^\times .

Lemma 3.2. [KP84, Proposition 0.1.5] *For $a, b \in E^\times$, let \tilde{a}, \tilde{b} be any of the inverse images of a, b in \tilde{E}^\times . The commutator $[\tilde{a}, \tilde{b}] \in \mu_2$ depends only on a, b , and is given by*

$$[\tilde{a}, \tilde{b}] = (a, b)_E (\mathbb{N}a, \mathbb{N}b)_F,$$

where $(\cdot, \cdot)_E$ and $(\cdot, \cdot)_F$ denote the quadratic Hilbert symbol of the field E and F respectively and $\mathbb{N} : E^\times \rightarrow F^\times$ is the norm map for the field extension E/F .

The following well-known relationship among the Hilbert symbols will be very useful to us. We thank Adrian Vasiu for the proof below. We will use this relationship on several occasions, sometimes without explicitly mentioning it.

Lemma 3.3. *Let E/E be a finite extension of p -adic fields with $\mu_n \subset F^\times$. For $a \in E^\times$ and $b \in F^\times$, we have*

$$(a, b)_E = (\mathbb{N}a, b)_F,$$

relating the n -th Hilbert symbols on F and E .

Proof. Observe the following commutative diagram from the local class field theory:

$$(5) \quad \begin{array}{ccc} E^\times & \xrightarrow{\sigma} & \text{Gal}(E^{ab}/E) \\ \downarrow \mathbb{N} & & \downarrow \text{res} \\ F^\times & \xrightarrow{\sigma} & \text{Gal}(F^{ab}/F) \end{array}$$

For $a \in E^\times$, or $a \in F^\times$, let $\sigma_a = \sigma(a)$ be the corresponding element of $\text{Gal}(E^{ab}/E)$, or $\text{Gal}(F^{ab}/F)$ as the case may be. By definition,

$$(a, b)_E = \frac{\sigma_a(b^{1/n})}{b^{1/n}},$$

therefore, we have

$$(\mathbb{N}a, b)_E = \frac{\sigma_{\mathbb{N}a}(b^{1/n})}{b^{1/n}}.$$

By the above commutative diagram we have

$$\sigma_a|_{F^{ab}} = \sigma_{\mathbb{N}a}$$

and then the proof of the lemma follows. \square

Now we describe some properties of the group \tilde{E}^\times .

Lemma 3.4. *The subgroup $\tilde{E}^{\times 2}$ is contained in the center of \tilde{E}^\times , and the subgroup $\tilde{F}^\times \tilde{E}^{\times 2}$ of \tilde{E}^\times is a maximal abelian subgroup of \tilde{E}^\times .*

Proof. From the commutator relation in Lemma 3.2, it is clear that $\tilde{E}^{\times 2}$ is contained in the center of \tilde{E}^\times . From the same commutator relation combined with Lemma 3.3, it follows that \tilde{F}^\times is abelian. Since $\tilde{E}^{\times 2}$ is contained in the center of \tilde{E}^\times , the subgroup $\tilde{F}^\times \tilde{E}^{\times 2}$ is abelian.

To prove that the subgroup $\tilde{F}^\times \tilde{E}^{\times 2}$ is maximal abelian, let $\tilde{a} \in \tilde{E}^\times$ commute with all the elements of $\tilde{F}^\times \tilde{E}^{\times 2}$. We need to prove that $\tilde{a} \in \tilde{F}^\times \tilde{E}^{\times 2}$.

Since $\tilde{E}^{\times 2}$ is contained in the center of \tilde{E}^\times , $[\tilde{a}, \tilde{b}] = 1$ for all $\tilde{b} \in \tilde{F}^\times \tilde{E}^{\times 2}$ is equivalent to $[\tilde{a}, \tilde{b}] = 1$ for all $\tilde{b} \in \tilde{F}^\times$. Using Lemma 3.2 and Lemma 3.3, for $b \in F^\times$, we have:

$$\begin{aligned} [\tilde{a}, \tilde{b}] = 1 & \iff (a, b)_E (\mathbb{N}a, \mathbb{N}b)_F = 1 \\ & \iff (\mathbb{N}a, b)_F (\mathbb{N}a, b^2)_F = 1 \\ & \iff (\mathbb{N}a, b)_F = 1. \end{aligned}$$

Therefore if $[\tilde{a}, \tilde{b}] = 1$ for all $b \in F^\times$, then $\mathbb{N}a \in F^{\times 2}$, and this is possible only if $a \in F^\times E^{\times 2}$ (observe that since $e/\bar{e} = e^2/(e\bar{e})$, $E^1 \subset F^\times E^{\times 2}$). This proves that $\tilde{F}^\times \tilde{E}^{\times 2}$ is a maximal abelian subgroup of \tilde{E}^\times . \square

Lemma 3.5. *The group $\tilde{E}^{\times 2}$ is equal to the center of \tilde{E}^\times .*

Proof. We already know that $\tilde{E}^{\times 2}$ is contained in the center of \tilde{E}^\times , and that $\tilde{F}^\times \cdot \tilde{E}^{\times 2}$ is a maximal abelian subgroup of \tilde{E}^\times . Let $f \in \tilde{F}^\times$ be in the centre of \tilde{E}^\times . It follows that,

$$(f, \mathbb{N}e)_F = 1, \forall e \in E^\times.$$

Since the Hilbert symbol is a non-degenerate bilinear form on $F^\times/F^{\times 2}$, and $\mathbb{N}E^\times$ is an index 2 subgroup of F^\times , the orthogonal complement of $\mathbb{N}E^\times$ (this is defined to be

the set of elements $a \in E^\times$ such that $(a, x) = 1$ for all $x \in \mathbb{N}E^\times$ must contain $F^{\times 2}$ as a subgroup of index 2.

Suppose that $E = F(\sqrt{d})$. Observe the identity:

$$X^2 - dY^2 + dY^2 = X^2.$$

By the definition of the Hilbert symbol, this means that

$$(d, \mathbb{N}e)_F = 1,$$

for all $e \in E^\times$. Since d is not a square in F^\times , it follows that the group generated by $F^{\times 2}$ and d inside F^\times , i.e. $\langle F^{\times 2}, d \rangle$, is the orthogonal complement on $\mathbb{N}E^\times$ for the Hilbert symbol of F .

It follows that if $f \in F^\times$ commutes with $E^{\times 2}$, then $f \in \langle F^{\times 2}, d \rangle$. Since d has a square root in E^\times by definition, it follows that $\langle f, E^{\times 2} \rangle = E^{\times 2}$. This proves that $\tilde{E}^{\times 2}$ is equal to the center of \tilde{E}^\times . \square

4. HEISENBERG GROUP AND ITS REPRESENTATIONS

The inverse images of tori (both split and non-split) of $GL_2(F)$ inside $\widetilde{GL}_2(F)$ are extensions of abelian groups by μ_2 , and may be called ‘Heisenberg groups’. Although Heisenberg groups are omnipresent in representation theory, we do not know a convenient reference for our use, so we have preferred to define them and prove some of their key properties that will be used throughout the paper.

Definition 4.1 (Heisenberg Group). *A group Σ with center $Z(\Sigma)$ with $\Sigma/Z(\Sigma)$ finite, and with $[\Sigma, \Sigma] \cong \mathbb{Z}/p\mathbb{Z} \subset Z(\Sigma)$ for some prime p , will be called a Heisenberg group.*

For such a group Σ , the quotient $\Sigma/Z(\Sigma)$ is clearly an abelian group, and the commutator map defines a bilinear form

$$(6) \quad B : \Sigma/Z(\Sigma) \times \Sigma/Z(\Sigma) \rightarrow \mathbb{Z}/p.$$

i.e., for $e_1, e_2 \in \Sigma/Z(\Sigma)$, we have $B(e_1, e_2) := [\tilde{e}_1, \tilde{e}_2]$ where \tilde{e}_1, \tilde{e}_2 are arbitrary lifts of e_1, e_2 in Σ . For an abelian group X , let \hat{X} denote the group of characters of X . Define a homomorphism $f_B : \Sigma/Z(\Sigma) \rightarrow \widehat{\Sigma/Z(\Sigma)}$ as follows: for all $a, e \in \Sigma/Z(\Sigma)$,

$$f_B(a)(e) := \exp \frac{2\pi i}{p} B(a, e).$$

Observe that the homomorphism f_B is injective (if $[\tilde{a}, \tilde{e}] = 1 \forall e \in \Sigma$, then by definition, $\tilde{a} \in Z(\Sigma)$). The bilinear form B is said to be non-degenerate if the corresponding homomorphisms f_B from $\Sigma/Z(\Sigma)$ to its character group is an isomorphism. In terms of the bilinear form B , a subgroup A of Σ is abelian if and only if the bilinear form B on $AZ(\Sigma)/Z(\Sigma)$ is identically zero. The subgroups of $\Sigma/Z(\Sigma)$ on which the bilinear form B is identically zero are called isotropic subgroups. It follows that a subgroup A of Σ containing $Z(\Sigma)$ is maximal abelian if and only if its image in $\Sigma/Z(\Sigma)$ is maximal isotropic. If A is isotropic, the natural map from Σ/A to the set of characters \hat{A} of A is surjective and if A is maximal isotropic, this map is an isomorphism. Note that if A is an abelian subgroup of Σ then the subgroup of Σ generated by A and $Z(\Sigma)$ is also abelian. It follows that if A is a maximal abelian subgroup of Σ then A necessarily contains the center $Z(\Sigma)$ of Σ .

Lemma 4.2. *Let Σ be a Heisenberg group in the sense of Definition 4.1 for which the corresponding bilinear form B given in (6) is non-degenerate. Let A be a maximal abelian subgroup of Σ , then*

$$[A : Z(\Sigma)]^2 = [\Sigma : Z(\Sigma)].$$

Proof. Because of the non-degeneracy of the bilinear form B , the natural map $\Sigma/A \rightarrow \widehat{A/Z(\Sigma)}$ is an isomorphism and hence $[\Sigma : A] = |\widehat{A/Z(\Sigma)}|$. The lemma follows from the obvious relations:

$$[\Sigma : A] \cdot [A : Z(\Sigma)] = [\Sigma : Z(\Sigma)] \text{ and } [A : Z(\Sigma)] = |\widehat{A/Z(\Sigma)}|.$$

□

A key property of Heisenberg groups that we will use is the following. Let A be a maximal abelian subgroup of Σ (thus containing $Z(\Sigma)$). Then Σ/A is an abelian group and naturally acts on \hat{A} , the set of characters of A which are non-trivial on $\mathbb{Z}/p \subset Z(\Sigma)$. This action of Σ/A on \hat{A} is faithful, i.e., $\chi^e \neq \chi$ for $e \neq 1$ in Σ/A . Equivalently, if $e \in \Sigma \setminus A$, $\chi \in \hat{A}$ is non-trivial on $\mathbb{Z}/p \subset Z(\Sigma)$, then there exists an $a \in A$ such that $\chi(eae^{-1}) \neq \chi(a)$, i.e. $\chi(eae^{-1}a^{-1}) \neq 1$. By the maximality of the abelian subgroup A inside Σ , given $e \in \Sigma \setminus A$ there exists an $a \in A$ such that $eae^{-1}a^{-1} \neq 1$. Since $eae^{-1}a^{-1} \in \mathbb{Z}/p$ and any nontrivial element of \mathbb{Z}/p generates \mathbb{Z}/p , it follows that for any nontrivial character χ of A which is nontrivial on \mathbb{Z}/p , $\chi(eae^{-1}a^{-1}) \neq 1$ for any $e \in \Sigma \setminus A$.

Definition 4.3. *An irreducible representation of a subgroup of a Heisenberg group Σ which contains $Z(\Sigma)$, the center of Σ , is called genuine if its restriction to $\mathbb{Z}/p \subset Z(\Sigma)$ is a non-trivial character of \mathbb{Z}/p .*

Proposition 4.4. *Let Σ_1 be a maximal abelian subgroup of a Heisenberg group Σ (such a subgroup is automatically normal and contains the center $Z(\Sigma)$ of Σ). Then*

- (1) *Any irreducible genuine representation of Σ is obtained by inducing a genuine character of Σ_1 .*
- (2) *Conversely, $\text{Ind}_{\Sigma_1}^{\Sigma} \lambda$ is irreducible for any character $\lambda : \Sigma_1 \rightarrow \mathbb{C}^\times$ with $\lambda|_{\mathbb{Z}/p} \neq 1$.*
- (3) *For characters $\lambda_1, \lambda_2 : \Sigma_1 \rightarrow \mathbb{C}^\times$, we have $\text{Ind}_{\Sigma_1}^{\Sigma} \lambda_1 \cong \text{Ind}_{\Sigma_1}^{\Sigma} \lambda_2$ if and only if $\lambda_1 = \lambda_2^s$ for some $s \in \Sigma$.*
- (4) *The restriction of an irreducible genuine representation of Σ to Σ_1 is a sum of distinct genuine characters $\lambda^s : \Sigma_1 \rightarrow \mathbb{C}^\times$, for $s \in \Sigma/\Sigma_1$.*
- (5) *The restriction of an irreducible genuine representation σ of Σ to Σ_1 is sum of all genuine characters of Σ_1 with multiplicity 1 whose restriction to $Z(\Sigma)$ is ω_σ , the central character of σ , i.e.,*

$$\sigma|_{\Sigma_1} = \text{ind}_{Z(\Sigma)}^{\Sigma_1} \omega_\sigma.$$

Proof. Let π be any irreducible genuine representation of Σ , and let λ be a character of Σ_1 which appears in π restricted to Σ_1 . By Clifford theory, if the action of Σ/Σ_1 on genuine characters of Σ_1 is faithful, then $\pi \cong \text{Ind}_{\Sigma_1}^{\Sigma} \lambda$ for any character λ of Σ_1 appearing in π . By the basic property of Heisenberg groups established already, we do know that the action of Σ/Σ_1 on genuine characters of Σ_1 is faithful proving part (1) and (2) of the proposition. Part (3) and (4) are clear as well. For part (5),

it is clear that $\text{ind}_{Z(\Sigma)}^{\Sigma_1} \omega_\sigma$ is contained in σ restricted to Σ_1 . To prove equality, it suffices to prove that the two representations have the same dimension. Observe that $\dim \text{ind}_{Z(\Sigma)}^{\Sigma_1} \omega_\sigma = \#(\Sigma_1/Z(\Sigma))$ where as by part (1), $\dim \sigma = \#(\Sigma/\Sigma_1)$. However by Lemma 4.2, $\#(\Sigma/Z(\Sigma)) = \#(\Sigma/\Sigma_1)^2$. Therefore, $\#(\Sigma/\Sigma_1) = \#(\Sigma/Z(\Sigma))$. \square

Corollary 4.5. *Suppose Σ_2 is an abelian subgroup of a Heisenberg group Σ with $\Sigma_2 Z(\Sigma)$ a maximal abelian in Σ . Then the restriction of an irreducible representation σ of Σ to Σ_2 is*

$$\text{ind}_{\Sigma_2 \cap Z(\Sigma)}^{\Sigma_2} \omega_\sigma.$$

5. REPRESENTATIONS OF \tilde{T} AND \tilde{E}^\times

To describe the representations of \tilde{T} and \tilde{E}^\times we first note the following fact.

Lemma 5.1. *The groups \tilde{T} and \tilde{E}^\times are Heisenberg groups in the sense of definition 4.1 with $[\tilde{T}, \tilde{T}] = \mathbb{Z}/2 = [\tilde{E}^\times, \tilde{E}^\times]$. Moreover, the bilinear forms corresponding to the Heisenberg groups \tilde{T} and \tilde{E}^\times , as defined in (6), are non-degenerate.*

Proof. Since T is abelian, $[\tilde{T}, \tilde{T}] \subset \mathbb{Z}/2$, and similarly E^\times being abelian, $[\tilde{E}^\times, \tilde{E}^\times] \subset \mathbb{Z}/2$. Since we know that \tilde{T} as well as \tilde{E}^\times are non-abelian (because we know they have a proper maximal abelian subgroup!), it follows that $[\tilde{T}, \tilde{T}] = \mathbb{Z}/2 = [\tilde{E}^\times, \tilde{E}^\times]$.

Now we prove the non-degeneracy of the bilinear forms for \tilde{T} and \tilde{E}^\times . For the rest of the proof, write Σ for either of the two Heisenberg groups \tilde{T} and \tilde{E}^\times . We need to prove that the homomorphism

$$(7) \quad \Sigma/Z(\Sigma) \rightarrow \widehat{\Sigma/Z(\Sigma)}$$

defined by $e \mapsto (x \mapsto [\tilde{e}, \tilde{x}])$, where \tilde{e} and \tilde{x} are arbitrary lift of e and x in Σ , is an isomorphism. Recall that the indices $[\tilde{T} : Z(\tilde{T})] = [\tilde{T} : \tilde{T}^2] = [F^\times : F^{\times 2}]^2$ and $[\tilde{E}^\times : \tilde{E}^{\times 2}] = [E^\times : E^{\times 2}]$ are finite. Since $\Sigma/Z(\Sigma)$ is a finite abelian group, the cardinality of $\Sigma/Z(\Sigma)$ and $(\widehat{\Sigma/Z(\Sigma)})$ are the same. Since the map (7) is known to be injective, it is also surjective. \square

Corollary 5.2. *For a maximal abelian subgroup A of \tilde{E}^\times (necessarily containing $\tilde{E}^{\times 2}$), we have $[\tilde{E}^\times : \tilde{E}^{\times 2}] = [\tilde{E}^\times : A]^2$. In particular, $[E^\times : E^{\times 2}] = [E^\times : F^\times E^{\times 2}]^2$.*

Remark 5.3. *Assume $p = 2$ for this remark. It is known that $[E^\times : E^{\times 2}] = 4 \cdot 2^{\deg(E/\mathbb{Q}_2)}$ and $[F^\times : F^{\times 2}] = 4 \cdot 2^{\deg(F/\mathbb{Q}_2)}$. For $E = F(\sqrt{d})$ we have $F^\times \cap E^{\times 2} = F^{\times 2} \cup dF^{\times 2}$ and hence $[F^\times E^{\times 2} : E^{\times 2}] = [F^\times : F^\times \cap E^{\times 2}] = \frac{1}{2}[F^\times : F^{\times 2}]$. Then the obvious identity $[E^\times : E^{\times 2}] = [E^\times : F^\times E^{\times 2}][F^\times E^{\times 2} : E^{\times 2}]$ confirms the conclusion in the above corollary.*

From the Proposition 4.4, Lemma 3.1 and Lemma 3.4, we deduce the following

Proposition 5.4. (1) *Up to isomorphism, an irreducible genuine representation σ of \tilde{T} is determined by its central character ω_σ (a character of \tilde{T}^2). If σ is an irreducible genuine representation of \tilde{T} with central character ω_σ , λ a character of $\tilde{Z}\tilde{T}^2$ which agrees with ω_σ when restricted to \tilde{T}^2 , the center of \tilde{T} , then $\sigma \cong \text{Ind}_{\tilde{Z}\tilde{T}^2}^{\tilde{T}} \lambda$.*

(2) *Up to isomorphism, an irreducible genuine representation of \tilde{E}^\times is determined by its central character. If σ is an irreducible genuine representation of \tilde{E}^\times with central character ω_σ , λ a character of $\tilde{F}^\times \tilde{E}^{\times 2}$ which agrees with ω_σ on $\tilde{E}^{\times 2}$, then $\sigma \cong \text{Ind}_{\tilde{F}^\times \tilde{E}^{\times 2}}^{\tilde{E}^\times} \lambda$.*

6. RESTRICTION OF PRINCIPAL SERIES REPRESENTATIONS

In this section, we study the restriction of a genuine principal series representation of $\widetilde{\mathrm{GL}}_2(F)$ to the subgroup \tilde{E}^\times for E a quadratic field extension of F .

We first recall the notion of a principal series representation. Let T, B and N be respectively the group of diagonal matrices, upper triangular matrices and upper triangular unipotent matrices in $G = \mathrm{GL}_2(F)$, and \tilde{T}, \tilde{B} and \tilde{N} their inverse images in the twofold cover $\tilde{G} = \widetilde{\mathrm{GL}}_2(F)$.

From the cocycle formula (2), it is clear that $\tilde{N} \cong N \times \mu_2$ and we identify N with $N \times \{1\}$ in $\widetilde{\mathrm{GL}}_2(F)$. One has $\tilde{B} = \tilde{T}N$.

Let τ be a genuine irreducible representation of \tilde{T} . Take the inflation of the representation τ to a representation of \tilde{B} by the quotient map $\tilde{B} \rightarrow \tilde{B}/N \cong \tilde{T}$ and denote this by the same letter τ . The induced representation $\mathrm{Ind}_{\tilde{B}}^{\tilde{G}} \tau$ of \tilde{G} is called a principal series representation of $\widetilde{\mathrm{GL}}_2(F)$.

Since \tilde{T} is a Heisenberg group, its genuine irreducible representations are determined (up to isomorphism) by its central character, i.e. a genuine character of \tilde{T}^2 . A character χ' of $F^{\times 2}$ can be considered as the restriction of a character χ of F^\times with $\chi'(a^2) = \chi^2(a)$. Two characters χ_1 and χ_2 of F^\times define the same character of $F^{\times 2}$ if and only if $\chi_1^2 = \chi_2^2$. Thus to a principal series representation of $\widetilde{\mathrm{GL}}_2(F)$, there is a naturally associated principal series representation $\chi_1^2 \times \chi_2^2$ of $\mathrm{GL}_2(F)$.

It is a theorem due to Moen [Moe89] that a principal series representation of $\widetilde{\mathrm{GL}}_2(F)$ is reducible if and only if the associated principal series representation $\chi_1^2 \times \chi_2^2$ of $\mathrm{GL}_2(F)$ is reducible.

We now study the restriction of π to the subgroup \tilde{E}^\times .

Lemma 6.1. *Let $\pi = \mathrm{Ind}_{\tilde{B}}^{\tilde{G}} \tau$ be a genuine principal series representation of $\widetilde{\mathrm{GL}}_2(F)$ with central character ω_π (a character of $\tilde{F}^{\times 2}$), i.e., $\omega_\tau|_{\tilde{F}^{\times 2}} = \omega_\pi$. Then the restriction of π to \tilde{E}^\times is*

$$\pi|_{\tilde{E}^\times} \cong \mathrm{Ind}_{\tilde{F}^{\times 2}}^{\tilde{E}^\times}(\omega_\tau) \cong \bigoplus_{\sigma \in \mathrm{Irrep}_{\omega_\pi}(\tilde{E}^\times)} (\dim \sigma) \sigma,$$

where $\mathrm{Irrep}_{\omega_\pi}(\tilde{E}^\times)$ denotes the set of isomorphism classes of irreducible genuine representations σ of \tilde{E}^\times such that $\omega_\sigma|_{\tilde{F}^{\times 2}} = \omega_\pi$. (Recall that by Lemma 3.5, center of \tilde{E}^\times is $\tilde{E}^{\times 2}$).

Proof. Note that the natural (right) action of \tilde{E}^\times on $\tilde{B} \backslash \tilde{G} = \mathbb{P}^1(F)$ is transitive, i.e. there is only one orbit. By Mackey theory, $\pi|_{\tilde{E}^\times} \cong \mathrm{Ind}_{\tilde{B} \cap \tilde{E}^\times}^{\tilde{E}^\times}(\tau|_{\tilde{B} \cap \tilde{E}^\times})$. Since $\tilde{B} \cap \tilde{E}^\times = \tilde{F}^\times$, we have $\pi|_{\tilde{E}^\times} \cong \mathrm{Ind}_{\tilde{F}^\times}^{\tilde{E}^\times}(\tau|_{\tilde{F}^\times})$. From Corollary 4.5,

$$\tau|_{\tilde{F}^\times} = \mathrm{Ind}_{\tilde{F}^\times \cap Z(\tilde{T})}^{\tilde{F}^\times}(\omega_\tau|_{\tilde{F}^\times \cap Z(\tilde{T})}) = \mathrm{Ind}_{\tilde{F}^{\times 2}}^{\tilde{F}^\times}(\omega_\pi).$$

Therefore, $\pi|_{\tilde{E}^\times} \cong \mathrm{Ind}_{\tilde{F}^{\times 2}}^{\tilde{E}^\times}(\omega_\pi)$. Since \tilde{E}^\times is a group which is compact modulo the center, the 2nd isomorphism in the assertion of the lemma follows from Frobenius reciprocity. \square

Corollary 6.2. *Let σ be an irreducible genuine representation of \tilde{E}^\times .*

- (1) Let $\pi = \mathrm{Ind}_{\tilde{B}}^{\tilde{G}} \tau$ be an irreducible genuine principal series such that $\omega_\sigma|_{\tilde{F}^\times 2} = \omega_\pi$. Then

$$\dim \mathrm{Hom}_{\tilde{E}^\times}(\pi, \sigma) = \dim \sigma = [E^\times : F^\times E^{\times 2}].$$

- (2) Let π and π' be the two sub-quotients of a genuine reducible principal series representation $\mathrm{Ind}_{\tilde{B}}^{\tilde{G}} \tau$ such that $\omega_\sigma|_{\tilde{F}^\times 2} = \omega_\pi$. Then for a quadratic extension E of F ,

$$\dim \mathrm{Hom}_{\tilde{E}^\times}(\pi, \sigma) + \dim \mathrm{Hom}_{\tilde{E}^\times}(\pi', \sigma) = [E^\times : F^\times E^{\times 2}].$$

7. RESTRICTION TO SPLIT TORUS

In this section, we study the restriction of an irreducible admissible genuine representation of $\widetilde{\mathrm{GL}}_2(F)$ to \tilde{T} . We will be utilizing the Kirillov model for the representations of $\widetilde{\mathrm{GL}}_2(F)$ [GPS80, Section 3].

Let $\psi : N \cong F \rightarrow \mathbb{C}^\times$ be a non-trivial character. Recall that the Kirillov model is an injective map $K : \pi \rightarrow C^\infty(F^\times, \pi_{N,\psi})$ such that the action of \tilde{B} on π is explicitly realized on the image of K . The subspace $\mathcal{S}(F^\times, \pi_{N,\psi})$ consisting of the functions which have compact support is contained in $K(\pi)$, the image of K , which is \tilde{B} -stable. Moreover, as a \tilde{B} -module we have the following short exact sequence

$$0 \rightarrow \mathcal{S}(F^\times, \pi_{N,\psi}) \rightarrow K(\pi) \rightarrow \pi_N \rightarrow 0.$$

Notice that $\pi_{N,\psi}$ is a \tilde{Z} -module on which N acts by the character ψ .

Proposition 7.1. *Let π be an irreducible admissible genuine representation of $\widetilde{\mathrm{GL}}_2(F)$.*

- (1) *As a \tilde{B} -module,*

$$\mathcal{S}(F^\times, \pi_{N,\psi}) \cong \mathrm{ind}_{\tilde{Z}N}^{\tilde{B}}(\pi_{N,\psi}).$$

- (2) *As a \tilde{T} -module,*

$$\mathcal{S}(F^\times, \pi_{N,\psi}) \cong \mathrm{ind}_{\tilde{Z}}^{\tilde{T}}(\pi_{N,\psi}).$$

Proof. The first part is part of Kirillov theory, see [GPS80, Section 3]., and the second is a consequence of Mackey theory. \square

Corollary 7.2. *Let π be an irreducible genuine supercuspidal representation of $\widetilde{\mathrm{GL}}_2(F)$ and σ an irreducible genuine representation of \tilde{T} with $\omega_\pi = \omega_\sigma|_{\tilde{Z}2}$. Then*

$$\mathrm{Hom}_{\tilde{T}}(\pi, \sigma) \cong \mathrm{Hom}_{\tilde{Z}}(\pi_{N,\psi}, \sigma) \cong \mathrm{Hom}_{\tilde{Z}2}(\pi_{N,\psi}, \omega_\sigma).$$

In particular,

$$\dim \mathrm{Hom}_{\tilde{T}}(\pi, \sigma) = \dim \pi_{N,\psi} = |\Omega(\pi, \psi)|$$

which is independent of the choice of the additive character ψ of F , and where

$$\Omega(\pi, \psi) = \{\omega : \tilde{F}^\times \rightarrow \mathbb{C}^\times \mid \pi_{N,\psi} \text{ contains } \omega_\pi\}.$$

Proof. Since $\pi_N = 0$, we have $\pi|_{\tilde{T}} \cong \mathcal{S}(F^\times, \pi_{N,\psi})|_{\tilde{T}} \cong \mathrm{ind}_{\tilde{Z}}^{\tilde{T}}(\pi_{N,\psi})$. From Corollary 4.5,

$$\sigma|_{\tilde{Z}} = \mathrm{Ind}_{\tilde{Z}2}^{\tilde{Z}}(\omega_\sigma).$$

By Frobenius reciprocity,

$$\mathrm{Hom}_{\tilde{T}}(\pi, \sigma) \cong \mathrm{Hom}_{\tilde{Z}}(\pi_{N,\psi}, \sigma) \cong \mathrm{Hom}_{\tilde{Z}2}(\pi_{N,\psi}, \omega_\sigma),$$

which proves the first isomorphism contained in the corollary.

The second isomorphism contained in the corollary follows since the multiplicity with which any genuine character of \tilde{Z} is contained in $\pi_{N,\psi}$ is exactly one [GHPS79, Theorem 4.1], and every character in $\Omega(\pi, \psi)$ restricted to \tilde{Z}^2 naturally equals $\omega_\pi = \omega_\sigma|_{\tilde{Z}^2}$. \square

Corollary 7.3. *Let π be an irreducible admissible genuine supercuspidal representation of $\widetilde{\mathrm{GL}}_2(F)$ and ψ a non-trivial additive character of F . Let π' be another finite length genuine supercuspidal representation of $\widetilde{\mathrm{GL}}_2(F)$ with the same central character as π , satisfying*

$$\Omega(\pi, \psi) \sqcup \Omega(\pi', \psi) = \Omega(\omega_\pi),$$

a disjoint union of sets, i.e., any character χ of \tilde{F}^\times whose restriction to $\tilde{F}^{\times 2}$ is ω_π appears in exactly one of $\pi_{N,\psi}$ or $\pi'_{N,\psi}$. (As recalled earlier, by [GHPS79, Theorem 4.1] every character of \tilde{F}^\times appearing in $\pi_{N,\psi}$ appears with multiplicity at most 1.)

Then we have

$$\pi|_{\tilde{T}} \oplus \pi'|_{\tilde{T}} \cong \bigoplus_{\sigma \in \mathrm{Irrep}_{\omega_\pi}(\tilde{T})} (\dim \sigma) \sigma$$

For a principal series representations $\mathrm{Ind}_{\tilde{B}}^{\tilde{G}} \tau$ where we use un-normalized induction, instead of using the Kirillov theory, we directly use the action of \tilde{T} on the principal series representation by the geometric nature of this action, and by Mackey theory get the following exact sequence of \tilde{T} -modules:

$$(8) \quad 0 \rightarrow \mathrm{ind}_{\tilde{Z}}^{\tilde{T}}(\tau|_{\tilde{Z}}) \rightarrow \pi \rightarrow \tau + \tau^\omega \rightarrow 0.$$

For an irreducible genuine representation σ of \tilde{T} , the functor $\mathrm{Hom}_{\tilde{T}}(-, \sigma)$ when applied to the short exact sequence in (8) results in the following long exact sequence:

$$0 \rightarrow \mathrm{Hom}_{\tilde{T}}(\tau + \tau^\omega, \sigma) \rightarrow \mathrm{Hom}_{\tilde{T}}(\pi, \sigma) \rightarrow \mathrm{Hom}_{\tilde{T}}(\mathrm{ind}_{\tilde{Z}}^{\tilde{T}}(\tau|_{\tilde{Z}}), \sigma) \rightarrow \mathrm{Ext}_{\tilde{T}}^1(\tau + \tau^\omega, \sigma)$$

From the Lemma 7.6 below, it follows that all representations of \tilde{T} except τ and τ^ω appear with the multiplicity with which it appears in $\mathrm{ind}_{\tilde{Z}}^{\tilde{T}}(\tau|_{\tilde{Z}})$ which is

$$\dim \mathrm{Hom}_{\tilde{T}}(\pi, \sigma) = [F^\times : F^{\times 2}].$$

On the other hand, if π is an irreducible principal series, it can also be expressed as $\mathrm{Ind}_{\tilde{B}}^{\tilde{G}}(\delta \cdot \tau^\omega)$ (as un-normalized induction). This realization of the principal series π gives us the following long exact sequence of \tilde{T} -modules:

$$(9) \quad 0 \rightarrow \mathrm{ind}_{\tilde{Z}}^{\tilde{T}}(\delta \tau^\omega|_{\tilde{Z}}) = \mathrm{ind}_{\tilde{Z}}^{\tilde{T}}(\tau^\omega|_{\tilde{Z}}) \rightarrow \pi \rightarrow \delta^{-1} \tau + \delta \tau^\omega \rightarrow 0.$$

Using this form of principal series, it follows by the same reasoning as above, that all representations of \tilde{T} except $\delta \cdot \tau^\omega$ and $\delta^{-1} \cdot \tau$ appear with the multiplicity with which it appears in $\mathrm{ind}_{\tilde{Z}}^{\tilde{T}}(\tau|_{\tilde{Z}})$ which is $[F^\times : F^{\times 2}]$.

Next we observe the following Lemma.

Lemma 7.4. *For an irreducible principal series representation $\pi = \mathrm{Ind}_{\tilde{B}}^{\tilde{G}} \tau$, and an irreducible genuine representation σ of \tilde{T} , either $\sigma \notin \{\tau, \tau^\omega\}$ or $\sigma \notin \{\delta \tau^\omega, \delta^{-1} \tau\}$.*

Proof. From Proposition 5.4(1), an irreducible representation σ of \tilde{T} is determined by its central character ω_σ (a character of \tilde{T}^2). Therefore, if $\sigma \in \{\tau, \tau^w\} \cap \{\delta\tau^w, \delta^{-1}\tau\}$, the central character of a representation in $\{\tau, \tau^w\}$ must be that of one in $\{\delta\tau^w, \delta^{-1}\tau\}$. Central character being a character of \tilde{T}^2 , write the central character of τ as (χ_1, χ_2) where $\chi_1, \chi_2 : F^{\times 2} \rightarrow \mathbb{C}^\times$. Thus the central character of τ^w is (χ_2, χ_1) , that of $\delta\tau^w$ is $(\nu^{1/2}\chi_2, \nu^{-1/2}\chi_1)$, and that of $\delta^{-1}\tau$ is $(\nu^{-1/2}\chi_1, \nu^{1/2}\chi_2)$. It follows that the set $\{\tau, \tau^w\} \cap \{\delta\tau^w, \delta^{-1}\tau\}$ is nonempty only when (as characters of $F^{\times 2}$)

$$(\chi_1/\chi_2, \chi_2/\chi_1) = (\nu^{1/2}, \nu^{-1/2}), \text{ or } (\nu^{-1/2}, \nu^{1/2}).$$

As recalled earlier, by [Moe89], these are exactly the conditions for reducibility of the genuine principal series representation $\text{Ind}_{\tilde{B}}^{\tilde{G}}\tau$, which we are excluding, thus the proof of the lemma is completed. \square

We summarize our analysis on irreducible principal series representations in the following proposition.

Proposition 7.5. *For an irreducible principal series representation $\pi = \text{Ind}_{\tilde{B}}^{\tilde{G}}\tau$, and an irreducible genuine representation σ of \tilde{T} such that $\omega_\pi = \omega_\sigma|_{\tilde{F}^{\times 2}}$,*

$$\dim \text{Hom}_{\tilde{T}}(\pi, \sigma) = [F^\times : F^{\times 2}].$$

Lemma 7.6. *Let σ_1 and σ_2 be two irreducible genuine representations of \tilde{T} with $\sigma_1|_{\tilde{Z}} = \sigma_2|_{\tilde{Z}}$. Then*

$$\dim \text{Hom}_{\tilde{T}}(\sigma_1, \sigma_2) = \dim \text{Ext}_{\tilde{T}}^1(\sigma_1, \sigma_2),$$

and $\text{Ext}_{\tilde{T}}^i(\sigma_1, \sigma_2) = 0$ for $i \geq 2$, where $\text{Ext}_{\tilde{T}}^i$ is calculated in the category of representations of \tilde{T} with a given central character of \tilde{Z}^2 which is $\sigma_1|_{\tilde{Z}^2} = \sigma_2|_{\tilde{Z}^2}$.

Proof. We know that any irreducible genuine representation of \tilde{T} is obtained as an induced representation from a genuine character of the finite index subgroup $\tilde{Z}\tilde{T}^2$, say $\sigma_2 = \text{ind}_{\tilde{Z}\tilde{T}^2}^{\tilde{T}}(\chi_2)$. By Frobenius reciprocity, for all $i \geq 0$ we have

$$\text{Ext}_{\tilde{T}}^i(\sigma_1, \sigma_2) = \text{Ext}_{\tilde{Z}\tilde{T}^2}^i(\sigma_1, \chi_2).$$

Since σ_1 restricted to the abelian subgroup $\tilde{Z}\tilde{T}^2$ is a sum of characters, we are reduced to the following claim whose proof we leave to the reader. (Our application of the claim below to the lemma above will involve $A = \tilde{Z}\tilde{T}^2$, and $C = \tilde{Z}^2$.)

Claim: If A is an abelian group with C a subgroup of A such that A/C is of the form $F \times \mathbb{Z}$, for F a pro-finite group. Then for characters χ_1 and χ_2 of A with $\chi_1|_C = \chi_2|_C$ one has $\text{Hom}_A(\chi_1, \chi_2) \cong \text{Ext}_A^1(\chi_1, \chi_2)$, and $\text{Ext}_A^i(\chi_1, \chi_2) = 0$ for $i \geq 2$ where Ext_A^i is calculated in the category of representations of A whose restriction to C is $\chi_1|_C = \chi_2|_C$. \square

Proposition 7.7. *Let π and π' be the two sub-quotients of a genuine reducible principal series representation $\text{Ind}_{\tilde{B}}^{\tilde{G}}\tau$, and σ an irreducible representation of \tilde{T} with $\omega_\sigma|_{\tilde{F}^{\times 2}} = \omega_\pi$. Then we have:*

$$\dim \text{Hom}_{\tilde{T}}(\pi, \sigma) + \dim \text{Hom}_{\tilde{T}}(\pi', \sigma) = [F^\times : F^{\times 2}],$$

except if σ is either π_N or π'_N .

Proof. The conclusion of the proposition follows from the exact sequence of Kirillov theory,

$$0 \rightarrow \mathcal{S}(F^\times, \pi_{N,\psi}) \rightarrow K(\pi) \rightarrow \pi_N \rightarrow 0,$$

together with Lemma 7.6. \square

Remark 7.8. *In the previous proposition, we do not know the exact value of $\dim \operatorname{Hom}_{\widetilde{T}}(\pi, \pi_N)$ which for all we know at the moment may take any of the two values $\dim \pi_{N,\psi}$ or $\dim \pi_{N,\psi} + 1$; similarly for $\dim \operatorname{Hom}_{\widetilde{T}}(\pi', \pi'_N)$. However, since $\dim \pi_{N,\psi}$ is either $[F^\times : F^{\times 2}] - 1$, or 1, this does not affect the conclusion of our main theorem 1.1.*

8. CORRESPONDENCE $P : \operatorname{Irrep}_{sc}(\widetilde{\operatorname{SL}}_2(F)) \rightarrow \operatorname{Irrep}_{sc}(\operatorname{SL}_2(F))$

Let $\operatorname{Irrep}_{sc}(\widetilde{\operatorname{SL}}_2(F))$ denote the set of isomorphism classes of irreducible genuine supercuspidal representations of $\widetilde{\operatorname{SL}}_2(F)$ and $\operatorname{Irrep}_{sc}(\operatorname{SL}_2(F))$, the set of isomorphism classes of irreducible supercuspidal representations of $\operatorname{SL}_2(F)$. Assuming that the residue characteristic p of the field F is odd, we shall define a natural correspondence from $\operatorname{Irrep}_{sc}(\widetilde{\operatorname{SL}}_2(F))$ to $\operatorname{Irrep}_{sc}(\operatorname{SL}_2(F))$. This correspondence will allow us to transfer a question on representations on the covering group $\widetilde{\operatorname{SL}}_2(F)$ to a similar question on the linear group $\operatorname{SL}_2(F)$. Let \mathcal{O}_F denote the ring of integers of F and ϖ a uniformizer in \mathcal{O}_F .

We recall the following well-known result.

Lemma 8.1. [Kub69] *Assume that the residue characteristic of F is odd. Let K be a maximal compact subgroup of $\operatorname{SL}_2(F)$. Then the covering $\widetilde{\operatorname{SL}}_2(F)$ of $\operatorname{SL}_2(F)$ splits when restricted to the subgroup K .*

Recall that there are two conjugacy classes of maximal compact subgroups of $\operatorname{SL}_2(F)$ which can be represented by

$$K_1 = \operatorname{SL}_2(\mathcal{O}_F) \text{ and } K_2 = \begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix} K_1 \begin{pmatrix} \varpi^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

We recall the following result due to Manderscheid.

Proposition 8.2. [Man84, Theorem 1.3] *Any irreducible supercuspidal representation of $\widetilde{\operatorname{SL}}_2(F)$ can be obtained as an induced representation from an irreducible finite dimensional representation of either \widetilde{K}_1 or \widetilde{K}_2 .*

Proposition 8.3. *Let K be a maximal compact subgroup of $\operatorname{SL}_2(F)$. There is a natural bijection between $\operatorname{Irrep}(K)$, the set of isomorphism classes of irreducible representations of K , and $\operatorname{Irrep}_{gen}(\widetilde{K})$ the set of isomorphism classes of irreducible genuine representations of \widetilde{K} ,*

$$\operatorname{Irrep}_{gen}(\widetilde{K}) \longleftrightarrow \operatorname{Irrep}(K).$$

Proof. Using a splitting $s : K \hookrightarrow \widetilde{G}$ given by Lemma 8.1, fix an isomorphism $\widetilde{K} \cong K \times_s \mu_2$. Observe that any two splittings $s_1, s_2 : K \hookrightarrow \widetilde{G}$ differ by a homomorphism $K \rightarrow \mu_2$. It is easy to see that K has no nontrivial characters of order 2 unless the residue field of F has order 2, 3. Since we are only considering odd residue characteristic anyway, and since it can be checked that the only nontrivial character of $\operatorname{SL}_2(\mathbb{F}_3)$ has order 3, there is a unique splitting in all the cases we are considering.

The isomorphism $\tilde{K} \cong K \times_s \mu_2$ defines a bijection between the set of isomorphism classes of irreducible representations of K , and irreducible genuine representations of \tilde{K} . Since there is a unique splitting over any maximal compact subgroup $SL_2(F)$, the bijection between irreducible representations of K and irreducible genuine representations of \tilde{K} is canonical. \square

Theorem 8.4. *Let $\text{Irrep}_{sc}(\widetilde{SL}_2(F))$ be the set of equivalence classes of irreducible admissible genuine supercuspidal representations of $\widetilde{SL}_2(F)$ and $\text{Irrep}_{sc}(SL_2(F))$, the set of equivalence classes of irreducible admissible supercuspidal representations of $SL_2(F)$. There is a natural bijection between*

$$P : \text{Irrep}_{sc}(\widetilde{SL}_2(F)) \rightarrow \text{Irrep}_{sc}(SL_2(F)).$$

Proof. Let $\tilde{\pi}$ be an irreducible admissible supercuspidal representation of $\widetilde{SL}_2(F)$. By the work of Manderscheid, Proposition 8.2, $\tilde{\pi}$ is isomorphic to an induced representation $\text{ind}_{\tilde{K}}^{\tilde{G}} \tilde{\sigma}$ which is induced from an irreducible representation $\tilde{\sigma}$ of a maximal compact subgroup \tilde{K} where K is either K_1 or K_2 . Let $\sigma \in \text{Irrep}(K)$ which corresponds to $\tilde{\sigma}$ in the manner described in Proposition 8.2, i.e., under the isomorphism $\tilde{K} = K \times \mathbb{Z}/2$, $\tilde{\sigma} = \sigma \times \text{sign}$.

We claim that $\pi := \text{ind}_K^G \sigma$ is an irreducible admissible supercuspidal representation of G . Given this claim, we define

$$P(\tilde{\pi}) = \pi.$$

It is known that if π is irreducible then it is also supercuspidal. So we shall only prove that π is irreducible. By [BH06, Theorem 3.11.4, and remark 1 following it] it suffices to prove that $\text{Hom}_G(\pi, \pi) = \mathbb{C}$ where we shorten the notation and write G for $SL_2(F)$. (We thank Sandeep Varma for this reference.) By Mackey theory, we have

$$\text{Hom}_G(\pi, \pi) = \mathbb{C} \oplus \left(\bigoplus_{1 \neq g \in K \backslash G / K} \text{Hom}_{K \cap K^g}(\sigma|_{K \cap K^g}, \sigma^g|_{K \cap K^g}) \right)$$

Note that under the natural map $\tilde{G} \rightarrow G$, the set of double cosets $\tilde{K} \backslash \tilde{G} / \tilde{K}$ is in bijection with the set of double cosets $K \backslash G / K$. For $g \in K \backslash G / K$, if we prove

$$\text{Hom}_{\tilde{K} \cap \tilde{K}^g}(\tilde{\sigma}|_{\tilde{K} \cap \tilde{K}^g}, \tilde{\sigma}^g|_{\tilde{K} \cap \tilde{K}^g}) \cong \text{Hom}_{K \cap K^g}(\sigma|_{K \cap K^g}, \sigma^g|_{K \cap K^g})$$

then theorem will follow, because the irreducibility of the compact induction for $\tilde{\pi} = \text{ind}_{\tilde{K}}^{\tilde{G}} \tilde{\sigma}$ implies that for $1 \neq g \in K \backslash G / K$, the space $\text{Hom}_{\tilde{K} \cap \tilde{K}^g}(\tilde{\sigma}|_{\tilde{K} \cap \tilde{K}^g}, \tilde{\sigma}^g|_{\tilde{K} \cap \tilde{K}^g}) = 0$. \square

Lemma 8.5. *There is an isomorphism*

$$\text{Hom}_{\tilde{K} \cap \tilde{K}^g}(\tilde{\sigma}|_{\tilde{K} \cap \tilde{K}^g}, \tilde{\sigma}^g|_{\tilde{K} \cap \tilde{K}^g}) \cong \text{Hom}_{K \cap K^g}(\sigma|_{K \cap K^g}, \sigma^g|_{K \cap K^g}).$$

In particular, both the spaces are simultaneously zero or non-zero.

Proof. Write $\tilde{K} \cong K \times_s \mu_2$ to emphasize the dependence of the isomorphism on the splitting s . Recall $\tilde{\sigma} = \sigma \otimes \text{sign}$. Note that any representative of the double coset $g \in K \backslash G / K$ can be regarded as a representative in $\tilde{K} \backslash \tilde{G} / \tilde{K}$.

If we know that the two isomorphisms $\tilde{K} \cap \tilde{K}^g \xrightarrow{\sim} (K \cap K^g) \times_s \mu_2$ and $\tilde{K} \cap \tilde{K}^g \xrightarrow{\sim} (K \cap K^g) \times_{s^g} \mu_2$ are the same then we have the following

$$\tilde{\sigma}|_{\tilde{K} \cap \tilde{K}^g} \cong \sigma|_{K \cap K^g} \otimes_s \text{sign} \text{ and } \tilde{\sigma}^g|_{\tilde{K} \cap \tilde{K}^g} \cong \sigma^g|_{K \cap K^g} \otimes_{s^g} \text{sign}.$$

Thus we have

$$\begin{aligned} \mathrm{Hom}_{\tilde{K} \cap \tilde{K}^g}(\tilde{\sigma}|_{\tilde{K} \cap \tilde{K}^g}, \tilde{\sigma}^g|_{\tilde{K} \cap \tilde{K}^g}) &\cong \mathrm{Hom}_{(K \cap K^g) \times \mu_2}(\sigma|_{K \cap K^g} \otimes \mathrm{sign}, \sigma^g|_{K \cap K^g} \otimes \mathrm{sign}) \\ &\cong \mathrm{Hom}_{K \cap K^g}(\sigma|_{K \cap K^g}, \sigma^g|_{K \cap K^g}). \end{aligned}$$

It remains to prove the following innocuous looking, but crucial lemma. \square

The following lemma uses that the covering group is the two fold cover of $\mathrm{SL}_2(F)$ since it crucially uses the fact that the inverse image in $\widetilde{\mathrm{SL}}_2(F)$ of the split torus in $\mathrm{SL}_2(F)$ is abelian, a property which is shared by all metaplectic covers, i.e., two fold covers of $\mathrm{Sp}_{2n}(F)$, but is not shared by general covers of general reductive groups.

Lemma 8.6. *Let $s : K \hookrightarrow \widetilde{\mathrm{SL}}_2(F)$ be a splitting and $g \in \mathrm{SL}_2(F)$. Let $s^g : K^g \hookrightarrow \widetilde{\mathrm{SL}}_2(F)$ be the splitting of K^g given by $s^g(k^g) = s(k)^{\tilde{g}}$ where \tilde{g} is any lift of g in $\widetilde{\mathrm{SL}}_2(F)$. Then*

$$s|_{K \cap K^g} = s^g|_{K \cap K^g}.$$

Proof. Note that:

- (1) It is enough to prove the lemma for $K = \mathrm{SL}_2(\mathcal{O}_F)$.
- (2) It is enough to prove the lemma for $g \in \mathrm{SL}_2(F)$ which are a set of representatives of the double cosets of K in $\mathrm{SL}_2(F)$. These representatives of the double coset of $K \backslash \mathrm{SL}_2(F) / K$ can be taken to be $\underline{a}_n := \begin{pmatrix} \varpi^n & 0 \\ 0 & \varpi^{-n} \end{pmatrix}$ for $n \in \mathbb{Z}_{\geq 0}$.

Note that the restriction of the two splittings s and s^g on $K \cap K^g$ differ by a character of $K \cap K^g$ with values in $\{\pm 1\}$, i.e., a quadratic character. Our aim is to prove that this character is trivial. This character on $K \cap K^g$ is given by

$$k \mapsto s(k^{-1})s^g(k) = s(k)^{-1}\tilde{g}s(k^g)\tilde{g}^{-1} \in \{\pm 1\}.$$

Let us write

$$\Gamma_0(m) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_F) : c \in \varpi^m \mathcal{O}_F \right\} \text{ if } m \in \mathbb{Z}_{\geq 0}$$

Then, for $g = \underline{a}_n$, the intersection $K \cap K^g = \Gamma_0(\varpi^{2n})$. For $n = 0$, there is nothing to prove. Now we assume $n \neq 0$ and, then $\Gamma_0(\varpi^{2n})$ has a normal pro- p subgroup $\Gamma_1(\varpi^{2n})$ with quotient isomorphic to $(\mathcal{O}_F/\varpi^{2n})^\times$. Since $p \neq 2$, a quadratic character of $K \cap K^g = \Gamma_0(\varpi^{2n})$ will factor through a quadratic character of $(\mathcal{O}_F/\varpi^{2n})^\times$. Note that this quadratic character of $\Gamma_0(\varpi^{2n})$ is trivial if it is trivial on the diagonal elements of $\Gamma_0(\varpi^{2n})$. Since the inverse image of the diagonal torus of $\mathrm{SL}_2(F)$ is abelian, the element $s(k)^{-1}\tilde{g}s(k^g)\tilde{g}^{-1} = s(k)^{-1}\tilde{g}s(k)\tilde{g}^{-1}$ is trivial for all diagonal k . Therefore, the map $k \mapsto s(k)^{-1}\tilde{g}s(k^g)\tilde{g}^{-1}$ is trivial. \square

Remark 8.7. *Since splitting $s : K \hookrightarrow \widetilde{\mathrm{SL}}_2(F)$ is unique, there is a natural way to write $\tilde{K} = s(K) \times \{\pm 1\} = K \times \{\pm 1\}$. Similarly, $\tilde{K}^g = K^g \times \{\pm 1\}$. An equivalent way to state the previous lemma would be that inside $\widetilde{\mathrm{SL}}_2(F)$, $(K \times \{\pm 1\}) \cap (K^g \times \{\pm 1\}) = (K \cap K^g) \times \{\pm 1\}$.*

The correspondence defined in Theorem 8.4 has the following striking property.

Proposition 8.8. *Let $\tilde{\pi}$ be an irreducible supercuspidal representation of $\widetilde{\mathrm{SL}}_2(F)$ and $\pi = P(\tilde{\pi})$ be the corresponding supercuspidal representation of $\mathrm{SL}_2(F)$. Then*

$$\tilde{\pi}|_K \cong \pi|_K.$$

Remark 8.9. *As mentioned before Lemma 8.6, all the results of this section (in particular, Theorem 8.4 and Proposition 8.8) are valid for the two fold metaplectic cover of $\mathrm{Sp}_{2n}(F)$ for F of odd residue characteristic.*

Remark 8.10. *If $p \neq 2$, the theorem on compact induction of irreducible supercuspidal representations of $\mathrm{SL}_2(F)$ as $\mathrm{ind}_K^{\mathrm{SL}_2(F)}(\sigma)$ allows one to construct an irreducible supercuspidal representation of $\widetilde{\mathrm{SL}}_2(F)$ as $\mathrm{ind}_{K \times \mu_2}^{\widetilde{\mathrm{SL}}_2(F)}(\sigma \otimes \mathrm{sign})$. It is natural to expect that this way we have constructed all irreducible supercuspidal representation of $\widetilde{\mathrm{SL}}_2(F)$ — which indeed would be a consequence of the theorem of Manderscheid, i.e. Proposition 8.2 — although we would like to think that it is a consequence of generalities (some kind of Plancherel theorem because their numbers and formal degrees are the same). The advantage of this method would be that it would be a much more general method of proving the theorem on compact induction of irreducible supercuspidal representations of other covering groups such as the two fold cover of $\mathrm{Sp}_{2n}(F)$. (It is known that the maximal reductive quotient of any maximal compact subgroup of $\mathrm{Sp}_{2n}(F)$ is of the form $\mathrm{Sp}_{2\ell} \times \mathrm{Sp}_{2m}$, in particular is simply connected, so the metaplectic covering of $\mathrm{Sp}_{2n}(F)$ when restricted to any maximal compact subgroup of $\mathrm{Sp}_{2n}(F)$ splits.) To be sure, our argument on irreducibility of $\mathrm{ind}_{K \times \mu_2}^{\widetilde{\mathrm{SL}}_2(F)}(\sigma \otimes \mathrm{sign})$ starting with the irreducible representation $\mathrm{ind}_K^{\mathrm{SL}_2(F)}(\sigma)$ of $\mathrm{SL}_2(F)$ works only for K a hyperspecial maximal compact subgroup of $\mathrm{Sp}_{2n}(F)$.*

9. RESTRICTION OF SUPERCUSPIDAL REPRESENTATIONS

9.1. Restriction of supercuspidal representations of $\widetilde{\mathrm{SL}}_2(F)$ to \tilde{E}^1 . In this subsection we study the restriction of an irreducible genuine supercuspidal representation of $\widetilde{\mathrm{SL}}_2(F)$ to a non-split torus. For any quadratic field extension E/F , let $E^1 := \{x \in E^\times : \mathbb{N}x = 1\}$ where \mathbb{N} denotes the norm of the quadratic extension. We fix an embedding $E^1 \hookrightarrow \mathrm{SL}_2(F)$ and we write E^1 for the non-split torus of $\mathrm{SL}_2(F)$ determined by the field extension E/F . Let \tilde{E}^1 denote the inverse image of E^1 in the twofold cover $\widetilde{\mathrm{SL}}_2(F)$.

Lemma 9.1. *The group \tilde{E}^1 is abelian.*

Proof. We already know that $E^1 \subset F^\times E^{\times 2}$, and $\tilde{F}^\times \tilde{E}^{\times 2}$ is a maximal abelian subgroup of \tilde{E}^\times , so the lemma follows. \square

Given a quadratic extension E/F , E^\times operates on the two dimensional vector space E over F , with E^1 leaving stable the maximal compact subring of E , thus $E^1 \subset K$, for K a maximal compact subgroup of $\mathrm{SL}_2(F)$. For $p \neq 2$, we know that the twofold cover $\widetilde{\mathrm{SL}}_2(F)$ splits over K and hence over E^1 giving rise to an isomorphism $\tilde{E}^1 \cong E^1 \times \{\pm 1\}$. For any genuine character $\tilde{\nu}$ of \tilde{E}^1 , we associate a character ν of E^1 such that $\tilde{\nu} = \nu \otimes \mathrm{sign}$.

The following result is a consequence of Proposition 8.8.

Proposition 9.2. *Let the residue characteristic of F be odd. Let $\tilde{\pi}$ be an irreducible genuine supercuspidal representation of $\widetilde{\mathrm{SL}}_2(F)$ and \tilde{v} a genuine character of \tilde{E}^1 . Let $\pi = P(\tilde{\pi})$ and v the corresponding character of E^1 . Then there is a natural isomorphism:*

$$\mathrm{Hom}_{\tilde{E}^1}(\tilde{\pi}, \tilde{v}) \cong \mathrm{Hom}_{E^1}(\pi, v).$$

In the odd residue characteristic case, Proposition 9.2 enables one to transfer the question of restriction of a supercuspidal representation of $\widetilde{\mathrm{SL}}_2(F)$ to \tilde{E}^1 to a similar question of restriction of a supercuspidal representation of $\mathrm{SL}_2(F)$ to E^1 .

9.2. Restriction of supercuspidal representations of $\widetilde{\mathrm{GL}}_2(F)$ to \tilde{E}^\times . In Proposition 9.2, we have transferred the restriction problem on covering groups to another restriction problem on linear groups where it is better understood. Since we are interested in understanding the restriction of an irreducible genuine supercuspidal representation of $\widetilde{\mathrm{GL}}_2(F)$ to \tilde{E}^\times , we now transfer this question to a related question of restriction of an irreducible genuine supercuspidal representation of $\widetilde{\mathrm{SL}}_2(F)$ to \tilde{E}^1 . We do this without any assumption on the residue characteristic of F .

Proposition 9.3. *Let $\tilde{\sigma}$ be an irreducible admissible genuine representation of $\widetilde{\mathrm{SL}}_2(F)$ and \tilde{v} a genuine character of \tilde{E}^1 . Let μ be a character of \tilde{Z} such that the restriction of μ to the center of $\widetilde{\mathrm{SL}}_2(F)$ is the central character of $\tilde{\sigma}$ and*

$$\tilde{\pi} = \mathrm{ind}_{\tilde{Z}\widetilde{\mathrm{SL}}_2(F)}^{\widetilde{\mathrm{GL}}_2(F)} \mu \tilde{\sigma} \cong \mathrm{ind}_{\tilde{Z}\widetilde{\mathrm{SL}}_2(F)}^{\widetilde{\mathrm{GL}}_2(F)} \mu^a \tilde{\sigma}^a.$$

Let λ be a genuine character of $\tilde{F}^\times \tilde{E}^{\times 2}$ such that $\lambda|_{\tilde{F}^\times} = \mu$ and $\lambda|_{\tilde{E}^1} = \tilde{v}$. Let us write $\tilde{\chi} = \mathrm{ind}_{\tilde{F}^\times \tilde{E}^{\times 2}}^{\tilde{E}^\times} \lambda$. Then

$$\mathrm{Hom}_{\tilde{E}^\times}(\tilde{\pi}, \tilde{\chi}) \cong \mathrm{Hom}_{\tilde{E}^1}(\tilde{\sigma}, \tilde{v}).$$

Proof. First observe that: $F^\times E^{\times 2} = F^\times E^1$ (since for $e \in E^\times$, $e^2 = (e\bar{e})(\frac{e}{\bar{e}}) \in F^\times E^1$, so $F^\times E^{\times 2} \subset F^\times E^1$; further, $\frac{e}{\bar{e}} = \frac{e^2}{e\bar{e}} \in F^\times E^{\times 2}$, so $F^\times E^{\times 2} \supset F^\times E^1$). Therefore $\tilde{F}^\times \tilde{E}^{\times 2} = F^\times E^1 \subseteq \tilde{Z}\widetilde{\mathrm{SL}}_2(F)$. From [PP16], recall the following

$$\tilde{\pi}|_{\tilde{Z}\widetilde{\mathrm{SL}}_2(F)} \cong \bigoplus_{a \in F^\times / F^{\times 2}} \mu^a \tilde{\sigma}^a.$$

Using Frobenius reciprocity, we get

$$\begin{aligned} \mathrm{Hom}_{\tilde{E}^\times}(\tilde{\pi}, \tilde{\chi}) &\cong \mathrm{Hom}_{\tilde{F}^\times \tilde{E}^{\times 2}}(\tilde{\pi}, \lambda) \\ &= \bigoplus_{a \in F^\times / F^{\times 2}} \mathrm{Hom}_{\tilde{F}^\times \tilde{E}^1}(\mu^a \tilde{\sigma}^a, \lambda). \end{aligned}$$

Recall that $\mu = \mu^a$ if and only if $a \in F^{\times 2}$. We are assuming that $\lambda|_{\tilde{F}^\times} = \mu$ and $\lambda|_{\tilde{E}^1} = \tilde{v}$, therefore we get

$$\begin{aligned} \mathrm{Hom}_{\tilde{E}^\times}(\tilde{\pi}, \tilde{\chi}) &\cong \mathrm{Hom}_{\tilde{F}^\times \tilde{E}^1}(\mu \tilde{\sigma}, \lambda) \\ &= \mathrm{Hom}_{\tilde{E}^1}(\tilde{\sigma}, \tilde{v}). \end{aligned}$$

This completes the proof of the proposition. \square

Remark 9.4. *It can be easily seen that for a given irreducible admissible genuine representation $\tilde{\pi}$ of $\widetilde{\mathrm{GL}}_2(F)$ and an irreducible genuine representation $\tilde{\chi}$ of \tilde{E}^\times with $\mathrm{Hom}_{\tilde{E}^\times}(\tilde{\pi}, \tilde{\chi}) \neq 0$, one can make suitable choices for an irreducible genuine supercuspidal representation $\tilde{\sigma}$ of $\widetilde{\mathrm{SL}}_2(F)$ and a genuine character \tilde{v} of \tilde{E}^1 which satisfies the conditions in Proposition 9.3. It*

follows from [PP16] that any irreducible genuine representation $\tilde{\pi}$ of $\widetilde{GL}_2(F)$ is of the form $\tilde{\pi} = \text{ind}_{\widetilde{SL}_2(F)}^{\widetilde{GL}_2(F)} \mu \tilde{\sigma}$ used in proposition 9.3.

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